Quasiclassical quantisation and radiative decay of sine-Gordon solitons pinned by a micro-inhomogeneity

1988 J. Phys. A: Math. Gen. 21 1553

(http://iopscience.iop.org/0305-4470/21/7/018)

J. Phys. A: Math. Gen. **21** (1988) 1553-1561. Printed in the UK

Quasiclassical quantisation and radiative decay of sine-Gordon solitons pinned by a micro-inhomogeneity

Yuri S Kivshar† and Boris A Malomed‡

† Institute for Low Temperature Physics and Engineering, 47 Lenin Avenue, Kharkov, 310164, USSR

‡ P P Shirshov Institute for Oceanology of the USSR Academy of Science, 23 Krasikov Street, Moscow, 117218, USSR

Received 9 September 1987

Abstract. Dynamics of a sine-Gordon kink and small-amplitude breather bound on an attractive localised inhomogeneity are studied. The rates of energy emission from the kink oscillating near the inhomogeneity and from a quiescent breather performing internal oscillations are calculated. In the semiclassical approximation, discrete quantised energy spectra for the solitons of both types are found, with regard to field-theory corrections to a kink's particle-like spectrum and to strong distortion of the breather's shape. The rates of radiative energy emission are interpreted as finite widths of the quantised quasiclassical energy levels. Oscillations of the breather as a whole near the inhomogeneity are also considered.

1. Introduction

At the present time, the dynamics of solitons in exactly integrable systems has been well studied, both in the classical (Zakharov *et al* 1980, Ablowitz and Segur 1981) and quantum (Faddeev and Korepin 1978, Rajaraman 1982) cases. However, in many physical applications there often occur equations with small Hamiltonian terms which break the exact integrability. There are a number of papers devoted to the development of a perturbative approach to such systems (see, e.g., McLaughlin and Scott 1978, Newell 1978, Kosevich and Kivshar 1982, Nozaki 1982, Olsen and Samuelsen 1982, Karpman *et al* 1983, Kivshar 1984, Malomed 1984, 1985, 1987a, b, Kivshar and Malomed 1985, Salerno *et al* 1985). However, most of these papers deal with adiabatic effects only, i.e. those disregarding radiative losses. In addition, the previous papers, except for Malomed (1987a, b), were devoted to the study of purely classical problems. In the present paper we intend to consider radiative and quantum (quasiclassical) effects in a system described by a perturbed sine-Gordon equation. As will be seen from the following, the methods we develop and use here are applicable to a rather wide class of various conservative (Hamiltonian) perturbations. However, in this work we will consider the perturbed system

$$u_{tt} - u_{xx} + \sin u = \varepsilon\delta(x)\sin u \qquad 0<\varepsilon \ll 1 \tag{1.1}$$

only, for which calculations are relatively simple, and results can be presented in a sufficiently complete form. Equation (1.1) has at least two interesting physical applications: it described a microresistor (provided $\varepsilon>0$) in a long Josephson junction (see, e.g., McLaughlin and Scott 1978) and a localised magnetic impurity ('light spin') in quasi-one-dimensional two-sublattice weak ferromagnets of the rare-earth orthoferrite type (see, e.g., Kivshar 1984).

The unperturbed sine-Gordon equation has soliton solutions of two types (see, e.g., Zakharov *et al* 1980, Ablowitz and Segur 1981): the kink

$$u_{\mathrm{k}}^{(0)} \approx 4 \tan^{-1}\{\exp[\sigma(x-\xi)]\} \tag{1.2}$$

and the breather

$$u_{\mathrm{br}}^{(0)} \approx 4\mu \cos[(1-\tfrac{1}{2}\mu^2)t] \operatorname{sech}[\mu(x-\xi)] \tag{1.3}$$

where ξ is the soliton's coordinate, its velocity $\dot{\xi}$ being assumed small: $\dot{\xi}^2 \ll 1$, $\sigma = \pm 1$ is the kink's polarity and μ is the breather's amplitude (we consider only the case of the small-amplitude breather, i.e. $\mu \ll 1$).

As is well known, the inhomogeneity with $\varepsilon > 0$ attracts a kink of either polarity (McLaughlin and Scott 1978), and a kink moving with sufficiently small velocity can be trapped by it on account of radiative energy losses (Malomed 1985). If we consider a trapped (pinned) kink, in the adiabatic approximation it moves as a non-relativistic particle with mass $m = 8$ in the potential (McLaughlin and Scott 1978)

$$U = -2\varepsilon \operatorname{sech}^2 \xi. \tag{1.4}$$

The corresponding law of motion can readily be found:

$$\sinh \xi(t) = [(2\varepsilon - \tilde{E})/\tilde{E}]^{1/2} \sin(\omega t) \tag{1.5}$$

where $\omega \equiv \sqrt{\tfrac{1}{2}\tilde{E}}$, $\tilde{E}$ being a parameter taking the values $0 < \tilde{E} \leqslant 2\varepsilon$.

In the same (particle-like) approximation, quasiclassical (WKB) quantisation of the kink's motion is straightforward, the levels of the energy $E \equiv \tilde{E}/\gamma$ being (see, e.g., Landau and Lifshitz 1974)

$$E_n^{(0)} = (1/\gamma)(\sqrt{2\varepsilon} - \tfrac{1}{4}n\gamma)^2 \tag{1.6}$$

where γ is the non-dimensional coupling constant (Rajaraman 1982) and, as is generally used in quantum field theory, $\hbar = 1$. In § 2 we calculate field-theoretical corrections to the quantum mechanical formula (1.6); these corrections arise from perturbation-induced corrections to the kink's form. In the same section we calculate the rate of energy emission from the oscillating pinned kink. From the quasiclassical viewpoint, the emission rate can be interpreted as a finite width of a quantised energy level.

In § 3 we consider the interaction of the small-amplitude breather (1.3) with the same inhomogeneity (1.1). Here we deal with the two small parameters, ε and μ. The case $\varepsilon \ll \mu$ is relatively simple (Malomed 1987a, b). In this paper we pay basic attention to the case $\varepsilon \sim \mu$ when the perturbation essentially distorts the breather's shape. Using an asymptotic method analogous to that developed by Kosevich and Kovalev (1974) and Eleonski *et al* (1984), we find the breather's form and the radiation wave field emitted by the breather. Then we accomplish quasiclassical quantisation of the breather. We consider both the quiescent breather and one performing small oscillations near the inhomogeneity.

2. A kink bound on the inhomogeneity

2.1. *Field corrections to the kink's quasiclassical spectrum*

The analogy between the kink and the particle, mentioned in the introduction, is not exact. It is violated if one takes into account perturbation-induced corrections to the kink's form (1.2):

$$u_k = u_k^{(0)} + u^{(1)}. \tag{2.1}$$

According to Kosevich and Kivshar (1983) and Kivshar (1984), the correction is determined by the general formula

$$u^{(1)} = -\frac{1}{\pi}\int_{-\infty}^{\infty} (\mathrm{d}\lambda/\lambda)(\lambda^2+\tfrac{1}{4})^{-1}(\lambda^2-\tfrac{1}{4}+\mathrm{i}\lambda\tanh z)b(\lambda, t)\exp[\mathrm{i}(\lambda-1/4\lambda)x] \tag{2.2}$$

provided $(\mathrm{d}\xi/\mathrm{d}t)^2 \ll 1$, where $z = x-\xi$, and $b(\lambda, t)$ is a complex coefficient which constitutes the continuous spectrum scattering data in terms of the inverse scattering transform (Ablowitz and Segur 1981, Zakharov *et al* 1980). For the perturbation (1.1) evolution of the function $B(\lambda, t) \equiv b(\lambda, t)\exp[\mathrm{i}(\lambda+1/4\lambda)t]$ is determined by the following equation:

$$\partial B(\lambda, t)/\partial t = -\tfrac{1}{4}\mathrm{i}\varepsilon(\lambda^2+\tfrac{1}{4})^{-2}\exp[\mathrm{i}(\lambda+1/4\lambda)t] \\ \times(\tanh\xi/\cosh\xi)[\lambda^2-\tfrac{1}{4}+\mathrm{i}\lambda\tanh\xi]. \tag{2.3}$$

It is natural to single out two qualitatively different parts of (2.3). The first one, to be called non-resonant, yields the oscillating part of $b(\lambda, t)$:

$$b(\lambda, t) = -\tfrac{1}{2}\varepsilon\sigma\lambda(\lambda^2+\tfrac{1}{4})^{-2}(\lambda^2-\tfrac{1}{4}+\mathrm{i}\lambda\tanh\xi)(\tanh\xi/\cosh\xi) \tag{2.4}$$

which determines the correction $u^{(1)}$ to the kink's form via (2.2):

$$u^{(1)}(x,\xi) = -\frac{\varepsilon\sigma\sinh\xi}{2\cosh z\cosh^3\xi}[\theta(x)(1-\mathrm{e}^{-z}\cosh z-\mathrm{e}^{-\xi}\cosh\xi+x) \\ +\theta(-x)(1-\mathrm{e}^{z}\cosh z-\mathrm{e}^{\xi}\cosh\xi-x)] \tag{2.5}$$

where $\theta(x)$ is the step function: $\theta(x)=1$ if $x>0$, and $\theta(x)=0$ if $x<0$. The second part of (2.3) yields a secular term. As we will demonstrate in the next subsection, following the line of works by Malomed (1984) and Kivshar (1984), it actually determines the emission power, i.e. the rate of energy emission.

The quasiclassical quantisation rule for the sine-Gordon wave field is (see, e.g., Rajaraman 1982)

$$\int_0^T \mathrm{d}t\int_{-\infty}^{+\infty}\mathrm{d}x\, u_t^2 = 2\pi\gamma n \tag{2.6}$$

$T = 4\pi/(\tilde{E})^{1/2}$ being the oscillation period. Inserting (2.5) into (2.6) and (1.2) into (2.1), and then (2.1) into (2.6), we obtain

$$\gamma n = 4(\sqrt{2\varepsilon}-\sqrt{\tilde{E}})+\tilde{E}^{3/2}F((2\varepsilon/\tilde{E})^{1/2}) \tag{2.7}$$

where

$$F(x) = 1-x+\tfrac{1}{4}x^3-(2/\pi)(4x^2-1)(x^2-1)^{1/2}+(6/\pi)x^3\log[x+(x^2-1)^{1/2}]. \tag{2.8}$$

Equation (2.7) can be rewritten in a form where the quantum mechanical spectrum (1.6) and the small correction to it are explicitly separated:

$$E_n = E_n^{(0)} + E_n^{(1)} \qquad E_n^{(1)} = \tfrac{1}{2}\gamma(E_n^{(0)})^2F((2\varepsilon/\gamma E_n^{(0)})^{1/2}). \tag{2.9}$$

2.2. Emission from an oscillating kink

Even if we neglect dissipation, which is always present in real physical systems, the kink's oscillations will fade due to emission of radiation (see, e.g., Malomed 1984). The basic formula for calculating the spectral density of the emission power $P(\lambda)$ had been set forward in the paper by Malomed (1984):

$$P(\lambda)=2\frac{1+4\lambda^2}{\pi\lambda^2\gamma}\operatorname{Re}\left(b(\lambda,t)\frac{\mathrm{d}b^*(\lambda,t)}{\mathrm{d}t}\right) \tag{2.10}$$

where $b(\lambda, t)$ is determined by the same formula (2.3). Contrary to the non-resonant part (2.4), the expression (2.10) has contributions from the resonant part only that, from the viewpoint of (2.3), is a secular term. After a rather lengthy calculation, the basic ingredient of which is expanding the RHS of (2.3) with regard to (1.5) into a Fourier series, we obtain the following expression for the main part of the emission power spectral density:

$$P(\lambda)=\sum_{n=0}^{\infty}\frac{n\varepsilon^2}{8\pi\gamma(1+4\lambda^2)}(2\varepsilon/\tilde{E})^{-3/2}B^{2n}\delta[\lambda+1/4\lambda-(2n+1)\omega] \tag{2.11}$$

where we have designated

$$B=(\sqrt{2\varepsilon}-\sqrt{\tilde{E}})/(\sqrt{2\varepsilon}+\sqrt{\tilde{E}}). \tag{2.12}$$

As we see from (2.11), the emission takes place on the radiation frequencies $\lambda+1/4\lambda=(2n+1)\omega$ with $n=[1/2\omega]+m$, $m=0,1,2,\ldots$, [ ] standing for the integer part. The eventual expressions for the corresponding powers are

$$P_m=(\sqrt{\varepsilon}/64\pi\gamma)\tilde{E}^{5/4}([1/\sqrt{\tilde{E}}]+m)(m-\{1/\sqrt{\tilde{E}}\})^{-1/2}B^{2(m+[1/\sqrt{\tilde{E}}])}. \tag{2.13}$$

Here $\{x\}\equiv x-[x]$. For the typical values $\tilde{E}\sim\varepsilon$ (not $\tilde{E}\ll\varepsilon$) the total power P_{tot} can be expressed in terms of the standard special function (see, e.g., Bateman and Erdelyi 1953) $\Phi(z,s,\alpha)=\Sigma_{n=0}^{\infty}z^n(\alpha+n)^{-s}$,

$$P_{\mathrm{tot}}=\sum_{m=0}^{\infty}P_m\approx\frac{\sqrt{\varepsilon}}{64\pi\gamma}\tilde{E}B^{2(1+[1/\sqrt{\tilde{E}}])}\Phi(B^2,\tfrac{1}{2},1-\{1/\sqrt{\tilde{E}}\}). \tag{2.14}$$

Since $B<1$, we may say that (2.14) is exponentially small in $\sqrt{\tilde{E}}$, i.e. actually in $\sqrt{\varepsilon}$.

Evidently, (2.13) and (2.14) are applicable provided $\gamma P_{\mathrm{tot}}\ll\tilde{E}\omega\sim\tilde{E}^{3/2}$. As is shown in figure 1, this condition fails in small vicinities of the points $1/\sqrt{\varepsilon}=\sqrt{2}(1-B)(1+B)^{-1}n$, n being large integer numbers, when the corresponding multiple frequencies of the kink's motion $2n\omega$ (see (1.5)) get too close to the spectral gap's edge $\omega_0=1$.

From the quasiclassical viewpoint formulated in the preceding subsection, equation (2.14) with $\tilde{E}=E_n^{(0)}\gamma$ gives the rate $\Gamma(n,m)$ of the radiative transition from the nth level to the level with number $n-2([1/\sqrt{\tilde{E}}]+m+1)\equiv n-\Delta n$:

$$\Gamma(n,m)=P_m/2([1/\sqrt{\tilde{E}}]+m+1)\omega\approx P_m. \tag{2.15}$$

The quasiclassical formula (2.15) is applicable provided $\Delta n\ll n$. It is easy to verify that for the present problem this condition means $E_n^{(0)}\gg\gamma/8\varepsilon$. Since $\tilde{E}<2\varepsilon$, the necessary condition for the applicability of the quasiclassical approach is $\varepsilon\gg\frac{1}{4}\gamma$.

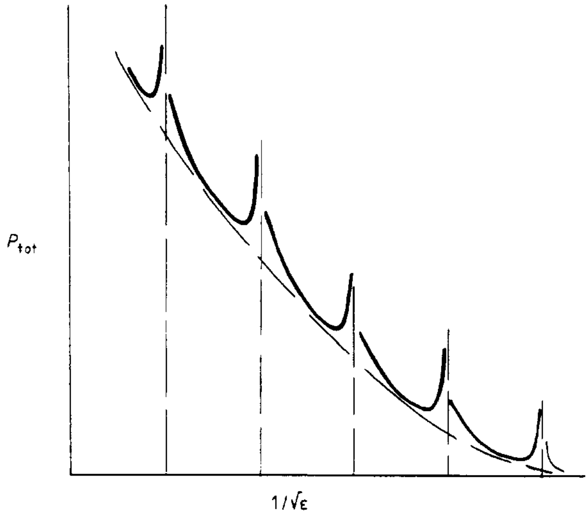

Figure 1. The dependence $P_{tot}(1/\sqrt{\varepsilon})$, B being fixed. The broken envelope curve is $P_{tot} = \text{constant} \times \varepsilon^{3/2} B^{\sqrt{2}(1+B)(1-B)^{-1}/\sqrt{\varepsilon}}$ (see equation (2.14)).

3. A small-amplitude breather interacting with the inhomogeneity

3.1. The shape of the perturbed breather

As was mentioned in the introduction, in the case $\mu \sim \varepsilon$ the shape of a small-amplitude breather is essentially different from (1.3). Indeed, according to (1.1), it is determined by the equation

$$u_{tt} - u_{xx} + \sin u = 0 \tag{3.1}$$

with the boundary conditions

$$u(x=+0) = u(x=-0) \qquad u_x(x=-0) - u_x(x=+0) = \varepsilon \sin u(x=0). \tag{3.2}$$

Following the lines of the asymptotic method (Kosevich and Kovalev 1974, 1975, Eleonski *et al* 1984), we seek a solution to (3.1) and (3.2) in the form of an expansion in powers of the small amplitude μ:

$$u(x, t) = A(x) \cos \omega t + B(x) \cos 3\omega t + \ldots \tag{3.3}$$

where

$$A(x) = \mu A_1(x) + \mu^3 A_3(x) + \ldots \qquad B(x) = \mu^3 B_3(x) + \ldots . \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.1) in the first approximation yields the relation

$$\omega \equiv (1-\mu^2)^{1/2} \approx 1 - \tfrac{1}{2}\mu^2 \tag{3.5}$$

(cf (1.3)), and the equation for $A_1(x)$:

$$\mathrm{d}^2 A_1/\mathrm{d}(\mu x)^2 - A_1 + \tfrac{1}{8}A_1^3 = 0. \tag{3.6}$$

The solution to (3.6) bounded at infinity is well known,

$$A_1(x) = 4\,\mathrm{sech}[\mu(x + x_0)] \tag{3.7}$$

x_0 being an arbitrary constant. Substituting (3.7) into (3.3), and (3.3) into the boundary condition (3.2), we obtain in the first approximation a solution of the form

$$u(x,t) \approx 4\mu \operatorname{sech}[\mu(|x|+x_0)] \cos \omega t \tag{3.8}$$

x_0 now being determined by the relation

$$\tanh(\mu x_0) = \varepsilon/2\mu. \tag{3.9}$$

In further approximations, substituting (3.3) and (3.4) into (3.1) enables us to express $A_3(x)$, $B_3(x), \ldots$, in terms of $A_1(x)$ (see Kosevich and Kovalev 1975). However, in the approximation $\sim\mu^3$ already, we cannot satisfy the boundary condition (3.2) since we have no more arbitrary parameters in the solution, if we admit only functions vanishing at $|x|\to\infty$. It is known (Kosevich and Kovalev 1975) that to resolve this problem one should supplement the solution (3.3) and (3.4) by travelling waves escaping from the breather. So, with accuracy up to the terms $\sim\mu^3$, the full solution takes the form

$$\begin{aligned} u(x,t) = 4\mu(\operatorname{sech} z)\{[1+\tfrac{1}{36}\mu^2(2-\operatorname{sech}^2 z)] \cos \omega t - \tfrac{1}{12}\mu^2 \operatorname{sech}^2 z \cos 3\omega t\} \\ + A_0 \cos(\sqrt{8}|x| - 3\omega t) \end{aligned} \tag{3.10}$$

ω being determined in (3.5), $z \equiv \mu(|x|+x_0)$ and

$$A_0 = -\tfrac{1}{3}\mu \tanh(\mu x_0) \operatorname{sech}^3(\mu x_0). \tag{3.11}$$

With this accuracy, the condition (3.9) for x_0 is replaced by

$$\tanh(\mu x_0) \approx (\varepsilon/2\mu)\{1+\mu^2[\tfrac{217}{18} - \tfrac{73}{6}(\varepsilon/2\mu)^2]\}. \tag{3.12}$$

The form of the 'distorted' breather (3.10)-(3.12) for the two cases $\varepsilon \gtrless 0$ is depicted in figures 2(a) and (b).

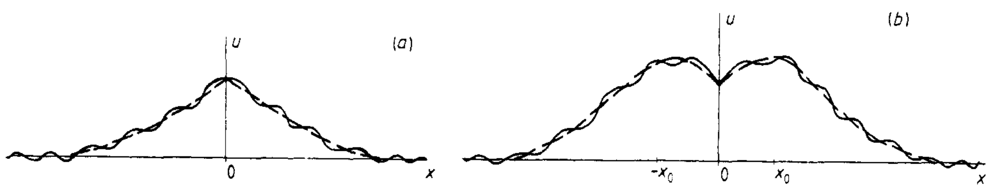

Figure 2. The shape of the breather (3.10)-(3.12); (a) $\varepsilon > 0$; (b) $\varepsilon < 0$. The broken curve corresponds to the first approximation (3.8) and (3.9). The full curve is wavy due to the radiation part of the wave field (the last term on the RHS of (3.10)).

It is natural to assume that the breather will be stable only in the case $\varepsilon > 0$, when the inhomogeneity attracts the wave field. The approach developed here for the case $\varepsilon \sim \mu$ is valid for $\varepsilon \ll \mu$ as well. One should keep in mind that equation (3.9) has no real solution, i.e. the breather does not exist, unless

$$\mu > \tfrac{1}{2}|\varepsilon|. \tag{3.13}$$

3.2. *Radiative decay and quantisation of the breather*

The energy E_{br} of the breather can be easily calculated in the approximation conformal to (3.8) and (3.9):

$$E_{br} = (16/\gamma)(\mu - \tfrac{1}{2}\varepsilon). \tag{3.14}$$

One can find the rate P of energy emission from the breather, using the next approximation (3.10)-(3.12):

$$P \equiv -\mathrm{d}E_{\mathrm{br}}/\mathrm{d}t = -\int_{-\infty}^{\infty} \mathrm{d}x \langle u_t u_x \rangle \tag{3.15}$$

the angular brackets standing for time averaging. The result is

$$P = \frac{3\times 625}{8\sqrt{2}\gamma}\, \varepsilon^2[\mu^2 - (\tfrac{1}{2}\varepsilon)^2]^3 \tag{3.16}$$

the radiation frequency being $\Omega = 3$.

In particular, in the limit case

$$0 < \nu \equiv \mu - \tfrac{1}{2}\varepsilon \ll \tfrac{1}{2}\varepsilon \tag{3.17}$$

when (3.8) takes the form

$$u(x, t) \approx 4\mu\, \mathrm{sech}[\mu|x| + \log(\varepsilon/\nu)] \cos \omega t \tag{3.18}$$

and, according to (3.14) and (3.15),

$$E_{\mathrm{br}} = 16\nu/\gamma \qquad P = (3\times 625/8\sqrt{2}\gamma)\varepsilon^5\nu^3 \tag{3.19}$$

i.e. ν plays the role of the effective breather's amplitude, the law of its fading, following from (3.19), is

$$\mathrm{d}\nu/\mathrm{d}t = -(3\times 625/128\sqrt{2})\varepsilon^5\nu^3. \tag{3.20}$$

As we see from (3.20), ν fades as $1/\sqrt{\varepsilon^5 t}$, provided $t \gg \varepsilon^{-7}$.

Finally, the breather can be quantised according to the Bohr–Sommerfeld rule (2.6). The final expression for the spectrum is

$$\mu_n = \mu_n^{(0)} + \mu_n^{(1)} \tag{3.21}$$

where $\mu_n^{(0)} = \frac{1}{16}\gamma n + \frac{1}{2}\varepsilon$, and

$$\mu_n^{(1)} = -\tfrac{1}{36}(\mu_n^{(0)})^3[\tfrac{80}{27} - \varepsilon/2\mu_n^{(0)} - \varepsilon^3/24(\mu_n^{(0)})^3]. \tag{3.22}$$

The expansion (3.21) is pertinent provided $1 \ll n \ll \mu^{-1}$. According to (3.14), the corresponding energy levels are

$$E_n = n + \mathrm{O}(\mu_n^{(0)3}). \tag{3.23}$$

Note that the term $\mu_n^{(0)}$ was found previously by Malomed (1987a, b) in the framework of the perturbation theory for solitons, i.e. for $\varepsilon \ll \mu$.

As to the emission power (3.16), from the quasiclassical viewpoint it gives the finite width of the discrete levels (3.23) stipulated by the possibility of the radiative transition $n \to n-3$, the transition rate being $\Gamma = \frac{1}{3}P$ (cf (2.15)).

3.3. Oscillations of the breather near the inhomogeneity

In § 3.2 we dealt with the quiescent breather. However, in the limit case (3.17) the asymptotic approach developed above can be generalised to describe small oscillations of the breather near the inhomogeneity. These oscillations will be called external to distinguish them from the internal oscillations of the breather. A corresponding solution $u(x, t)$ is looked for in the form (cf (3.18))

$$u(x, t) = 4\mu_j\, \mathrm{sech}(\mu_j|x| + \phi_j) \cos \omega t \qquad j = 1, 2 \tag{3.24}$$

where $j=1$ for $x>0$ and $j=2$ for $x<0$. Inserting (3.24) into the boundary conditions (3.2), we readily obtain the equations

$$\mu_1 \operatorname{sech} \phi_1 = \mu_2 \operatorname{sech} \phi_2 \tag{3.25a}$$

$$\mu_1 \tanh \phi_1 + \mu_2 \tanh \phi_2 = \varepsilon \tag{3.25b}$$

where we have set $\sin u(x=0) \approx u(x=0)$. The system of two linear equations (3.25) for μ_j yields

$$\mu_j = \varepsilon \cosh \phi_j / (\sinh \phi_1 + \sinh \phi_2). \tag{3.26}$$

Small external oscillations of the breather near the inhomogeneity can be described by setting $\phi_{1,2} = \phi \pm \Psi(t)$, $\Psi \ll \phi$, where ϕ is a constant and Ψ is a slowly varying function of time. Inserting this into (3.26), we obtain the expansions for μ_j:

$$\mu_{1,2} - \tfrac{1}{2}\varepsilon = \pm\tfrac{1}{2}\varepsilon \tanh \Psi + \varepsilon(1 \pm \Psi)\,\mathrm{e}^{-2\phi} + \mathrm{O}(\varepsilon \Psi^3 \mathrm{e}^{-2\phi}) \tag{3.27}$$

(recall that, due to (3.17), $\exp(-2\phi) \approx (\mu - \frac{1}{2}\varepsilon)\varepsilon^{-1}$ is a small quantity).

The underlying equation (1.1) corresponds to the following Lagrangian expanded in powers of the wave field up to u^4 terms:

$$L = \tfrac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x\,(u_t^2 - u_x^2 - u^2 + \tfrac{1}{12}u^4 + \varepsilon u^2 \delta(x)). \tag{3.28}$$

Inserting (3.24), (3.27) and (3.5) into (3.28) in the first approximation yields the following expression for the effective potential energy $U(\Psi)$ of the small external oscillations averaged in the fast internal oscillations:

$$U(\Psi) = 12\varepsilon^3 \Psi^2 \exp(-4\phi). \tag{3.29}$$

To find the effective kinetic energy T, we note that differentiating (3.27) yields, in the lowest approximation, $\mathrm{d}\mu/\mathrm{d}t \approx \frac{1}{2}\varepsilon\, \mathrm{d}\Psi/\mathrm{d}t$. Inserting this into (3.28) brings us to the averaged expression

$$T \approx 4\varepsilon \exp(-2\phi)(\mathrm{d}\Psi/\mathrm{d}t)^2. \tag{3.30}$$

Using (3.29) and (3.30), we immediately find the frequency χ of the small external oscillations:

$$\chi^2 = 3\varepsilon^2 \exp(-2\phi). \tag{3.31}$$

Since, according to (3.27), $\exp(-2\phi) \approx (\mu - \frac{1}{2}\varepsilon)\varepsilon^{-1}$, where μ stands for the mean value of μ_j, which is the same for $j=1$ and 2, we may finally write (3.31) in the form

$$\chi^2 = 3\varepsilon(\mu - \tfrac{1}{2}\varepsilon). \tag{3.32}$$

It is important that, due to (3.17), $\chi^2 \ll \varepsilon^2$. Indeed, the above consideration actually implied $\mu_j(t)$ and $\phi_j(t)$ to vary in time adiabatically slowly on the background of the fast internal oscillations. One can verify that, with the accuracy to which the expressions (3.29) and (3.30) have been derived, this assumption is warranted by the condition $\chi^2 \ll \varepsilon^2$. This is why the consideration of the small external oscillations can be performed self-consistently for the limit case (3.17) only.

From the quasiclassical viewpoint, the external oscillations result in the splitting of each internal oscillation level (3.23) into a band of 'fine structure' levels

$$E_{nm} = n + m\chi \tag{3.33}$$

where m is the quantum number of the external oscillations (in the quasiclassical case $m \gg 1$). Accordingly, each emission line corresponding to the radiative transition $n \to n-3$ between the internal oscillation levels will acquire 'fine structure' corresponding to the radiation frequencies $3+\Delta m\chi$, each line of the 'fine structure' being generated by a transition $(n, m) \to (n-3, m-\Delta m)$ between the split levels (3.33). These transitions may be considered quasiclassically provided $\Delta m \ll m$.

References

Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)
Bateman H and Erdelyi A 1953 *Higher Transcendental Functions* vol 1 (New York: McGraw-Hill)
Eleonski V M, Kulagin N E, Novozhilova N S and Silin V P 1984 *Teor. Mat. Fiz.* (*Sov. Phys.-Theor. Math. Phys.*) **60** 395
Faddeev L D and Korepin V E 1978 *Phys. Rep.* **42** 3
Karpman V I, Maslov E M and Solov'ev V V 1983 *Zh. Eksp. Teor. Fiz.* **84** 289
Kivshar Yu S 1984 *Institute for Low Temperature Physics and Engineering* (*Kharkov*) *preprint* (in Russian)
Kivshar Yu S and Malomed B A 1985 *Phys. Lett.* **111A** 427
Kosevich A M and Kivshar Yu S 1982 *Sov. J. Low Temp. Phys.* **8** 644
—— 1983 *Phys. Lett.* **98A** 237
Kosevich A M and Kovalev A S 1974 *Sov. Phys.-JETP* **40** 891
—— 1975 *Sov. J. Low Temp. Phys.* **1** 712
Landau L D and Lifshitz E M 1974 *Quantum Mechanics* (Moscow: Nauka)
Malomed B A 1984 *Phys. Lett.* **102A** 83
—— 1985 *Physics* **15D** 385
—— 1987a *Phys. Lett.* **120A** 28
—— 1987b *Physica* **27D** 113
McLaughlin D W and Scott A C 1978 *Phys. Rev.* A **18** 1652
Newell A C 1978 *J. Math. Phys.* **19** 1126
Nozaki K 1982 *Phys. Rev. Lett.* **49** 1883
Olsen O H and Samuelsen M R 1982 *Wave Motion* **4** 29
Rajaraman R 1982 *An Introduction to Solitons and Instantons in Quantum Field Theory* (Amsterdam: North-Holland)
Salerno, M, Samuelsen M R, Lomdahl P S and Olsen O H 1985 *Phys. Lett.* **108A** 241
Zakharov V E, Manakov S V, Novilov S P and Pitaevskii L P 1980 *Theory of Solitons. The Inverse Scattering Method* (Moscow: Nauka)